LAST TIME DEFINITION OF CRYSTALS OMITTED AN AXIOM. (I'LL CORRECT.)

EVEN WHEN THIS DEFINITION IS CORRECTED THERE FURTHER MORE COMPLICATED AXIOMS CALLED STEMBRIDGE AXIOMS NEEDED TO PIN DOWN THE "RIGHT" CRYSTALS. (TOPIC FOR LATER.)

DEFINITIONS. FOR $GL(n, \mathbb{C})$ CRYSTALS.

$T \subset GL(n, \mathbb{C})$ DIAGONAL SUBGROUP

$\Lambda = \mathbb{Z}^n$ "WEIGHT LATTICE" EVERY ELT OF $\mathbb{Z}^n$ IS CONSIDERED A CHARACTER OF $t$: $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$ $\lambda_i \in \mathbb{Z}$

$$t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \in T$$

$$t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n}.$$

IF WE RESTRICT A REP'N $(\pi, V)$ OF $GL(n, \mathbb{C})$ TO $T$ IT DECOMPOSES INTO CHARACTERS OF $T$ WITH MULTIPLICITIES

$$\chi_\pi(t) := tr\ \pi(t) = \sum_{\mu \in \Lambda} a_\mu t^\mu$$

$a_\mu$ IS A NONNEGATIVE INTEGER.

BECAUSE IF $\sigma \in S_n \hookrightarrow GL(n, \mathbb{C})$

$$\chi_\pi(\sigma(t)) = \chi_\pi(\sigma t \sigma^{-1}) = \chi_\pi(t)$$

THE POLYNOMIAL $\chi_\pi(t)$ $\left( \text{IN } t_1, \dots, t_n \text{ AND PERHAPS } \det(t)^{-1} \right)$ IS SYMMETRIC.

IF $\pi$ IS IRREDUCIBLE $\chi_\pi(t)$ IS KNOWN TO BE A SCHUR POLYNOMIAL (POSSIBLY DIVIDED BY $\det$ TO SOME POWER.)

EXAMPLE WHERE $\det(t)^{-1} = (t_1 \cdots t_n)^{-1}$

IS NEEDED IS CONTRAGREDIENT $\hat{\pi}$

$$GL(n, \mathbb{C}) \longrightarrow GL(n, \mathbb{C})$$

$$g \longrightarrow {}^t g^{-1} = \hat{\pi}(g)$$

$$\chi_{\hat{\pi}}(t) = t_1^{-1} + t_2^{-1} \cdots + t_n^{-1}$$

$$= \det(t)^{-1} (t_2 \cdots t_n + t_1 t_3 \cdots t_n + \cdots)$$

$$= \det(t)^{-1} e_{n-1}(t_1, \dots, t_n)$$

WE CAN AVOID $\det(t)^{-1}$ HERE

BY INSTEAD CONSIDERING

$$\det \otimes \hat{\pi} \cong \Lambda^{n-1} \mathbb{C}^n.$$

GL(3, $\mathbb{C}$) REPS FOR EXAMPLES.

| $\pi$ | $\chi_\pi\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}$ |
|---|---|
| STANDARD | $t_1 + t_2 + t_3 = \Delta_{(1,0,0)}$ |
| $\wedge^2$ | $t_1t_2 + t_2t_3 + t_1t_3 = \Delta_{(1,1,0)}$ |
| $V^2$ | $t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_2t_3 + t_1t_3 = \Delta_{(2,0,0)}$ |
| ADJOINT SQUARE | $\Delta_{(2,1,0)}$ SEE BELOW |

LET G $\circlearrowleft$ 3×3 MATRICES WITH TRACE ZERO BY CONJUGATION.

THIS IS THE ADJOINT REP'N. Ad

$$\chi_{Ad}\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} = t_1t_2^{-1} + t_1t_3^{-1} + t_2t_3^{-1} + t_2t_1^{-1} + t_3t_1^{-1} + t_3t_2^{-1} + 2$$

CLEAR DENOMINATORS BY TENSORING WITH det. THIS IS THE ADJOINT SQUARE REP'N

ITS CHARACTER IS

$$\Delta_{(2,1,0)} = t_1^2 t_2 + t_1^2 t_3 + t_1^2 t_2 + t_2^2 t_1 + t_3^2 t_1 + t_3^2 t_2 + 2 t_1 t_2 t_3$$

"SIMPLE ROOTS" $\alpha_1, \dots, \alpha_{n-1} \in \Lambda$

$\alpha_1 = (1,-1,0,\dots,0)$ $\quad \alpha_2 = (0,1,-1,0,\dots)$

$$\alpha_i = (0,\dots,0,\underset{\uparrow i}{1},-1,0,\dots)$$

<u>DEFINITION</u>: A SEMINORMAL CRYSTAL IS A SET $\mathcal{C}$ WITH A MAP

$$wt : \mathcal{C} \to \Lambda$$

AND MAPS $e_i, f_i : \mathcal{C} \to \mathcal{C} \cup \{0\}$

SATISFYING CERTAIN PROPERTIES

$$wt(e_i(x)) = wt(x) + \alpha_i \quad \text{IF } e_i(x) \neq 0$$

$$wt(f_i(x)) = wt(x) - \alpha_i \quad \text{IF } f_i(x) \neq 0$$

IF $e_i(x) = y \iff f_i(y) = x$.

DEFINE $\varepsilon_i(x)$ = # OF TIMES WE MAY APPLY $e_i$ TO $x$ I.E.

$\varepsilon_i(x) = n$ $\quad e_i^n(x) \neq 0$ BUT $e_i^{n+1}(x) = 0$

SIMILARLY $\varphi_i(x)$ = # OF TIMES WE MAY APPLY $f_i$. THEN

$$\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i^\vee \rangle$$

$\alpha_i^\vee = \dfrac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} = \alpha_i$ FOR $GL(n)$ CRYSTALS.

DISTINCTION BETWEEN $\alpha_i, \alpha_i^\vee$ NEEDED FOR CRYSTALS OF $Sp(2n)$ OR $O(2n+1)$ NOT FOR $GL(n, \mathbb{C})$

LAST PART OF DEFINITION WAS OMITTED LAST TIME.

EXAMPLES TO ILLUSTRATE:

STANDARD CRYSTAL,

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \boxed{4} \to \cdots \xrightarrow{n-1} \boxed{n}$$

$x \xrightarrow{i} y$ MEANS $f_i(x) = y$

$wt\left(\boxed{i}\right) = \mathbf{e}_i$ ($i$-TH BASIS VECTOR OF $\mathbb{Z}^n$)

$$= (0, \dots, 0, 1, 0, \dots)$$

$i$-TH POSITION

$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$

CHECK THE AXIOM $\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i \rangle$

TAKE $n = 3$.

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

$e_1(\boxed{1}) = 0$ $\qquad$ $e_2(\boxed{1}) = 0$

$e_1(\boxed{2}) = \boxed{1}$ $\qquad$ $e_2(\boxed{2}) = 0$

$e_1(\boxed{3}) = 0$ $\qquad$ $e_2(\boxed{3}) = \boxed{2}$

$\overset{x}{\bullet} \xrightarrow{i} \overset{y}{\bullet}$ MEANS $f_i(x) = y$ OR

EQUIV. $e_i(y) = x$

$$\underset{a}{\bullet} \xrightarrow{i} \underset{b}{\bullet} \xrightarrow{i} \underset{c}{\bullet} \xrightarrow{i} \underset{d}{\bullet} \xrightarrow{i} \underset{e}{\bullet}$$

$\varepsilon_i(b) = 1$ SINCE $e_i(b) = a$

$e_i(a) = 0$

$\varphi_i(b) = 3$ $\quad$ $f_i(b) = c$

$f_i(c) = d$

$f_i(d) = e$ $\qquad$ $f_i(e) = 0$

$e_1(\boxed{1}) = 0$ $\quad$ $e_2(\boxed{1}) = 0$ $\quad$ $f_i(x) = y \iff e_i(y) = x$

$e_1(\boxed{2}) = \boxed{1}$ $\quad$ $e_2(\boxed{2}) = 0$ $\quad$ $\alpha_1 = (1,-1,0)$

$e_1(\boxed{3}) = 0$ $\quad$ $e_2(\boxed{3}) = \boxed{2}$ $\quad$ $\alpha_2 = (0,1,-1)$

| $x$ | $wt(x)$ | $\varepsilon_1(x)$ | $\varphi_1(x)$ | $\langle wt(x), \alpha_1 \rangle$ |
|---|---|---|---|---|
| $\boxed{1}$ | $(1,0,0)$ | 0 | 1 | 1 |
| $\boxed{2}$ | $(0,1,0)$ | 1 | 0 | -1 |
| $\boxed{3}$ | $(0,0,1)$ | 0 | 0 | 0 |

AXIOM IS SATISFIED IN THIS CASE

$\alpha_2 = (0,1,-1)$

| $x$ | $wt(x)$ | $\varepsilon_2(x)$ | $\varphi_2(x)$ | $\langle wt(x), \alpha_2 \rangle$ |
|---|---|---|---|---|
| $\boxed{1}$ | $(1,0,0)$ | 0 | 0 | 0 |
| ✓ $\boxed{2}$ | $(0,1,0)$ | 0 | 1 | 1 |
| $\boxed{3}$ | $(0,0,1)$ | 1 | 0 | -1 |

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

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ACTION OF $S_n$ ON $\Lambda$

IS BY PERMUTING COMPONENTS.

$$S_n = \langle s_1, \dots, s_{n-1} \rangle$$

$s_i = (i, i+1)$ TRANSPOSITION.

FOR $\mu \in \Lambda$

$$s_i(\mu) = \mu - \langle \mu, \alpha_i \rangle \alpha_i$$

$$\langle \mu, \alpha_i \rangle = \mu_i - \mu_{i+1} \qquad \alpha_i = (0, \dots, \underset{\substack{\uparrow \\ i}}{1}, -1, 0, \dots)$$

$$\mu - \langle \mu, \alpha_i \rangle \alpha_i =$$

$$(\mu_1, \dots, \mu_i, \mu_{i+1}, \dots) - (\mu_i - \mu_{i+1})(0, \dots, 1, -1, 0, \dots)$$

$$= (\mu_1, \dots, \mu_{i+1}, \mu_i, \mu_{i+2}, \dots) = s_i(\mu)$$

THE MEANING OF

$$\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i \rangle$$

IS THAT WE CAN DEFINE AN ACTION OF $\Delta_i$ ON THE CRYSTAL AS FOLLOWS.

$$f_i^{\ell}(x) \leftarrow \cdots \leftarrow f_i(x) \leftarrow x \xleftarrow{i} e_i(x) \xleftarrow{i} \cdots \xleftarrow{i} e_i^{k}(x) \qquad (*)$$

ENDPOINTS

$k = \varepsilon_i(x) = \#$ OF TIMES WE CAN APPLY $e_i$

$\ell = \varphi_i(x)$

IF WE ERASE ALL ARROWS EXCEPT $i$-ARROWS FROM CRYSTAL GRAPH WHAT IS LEFT MAY BE DISCONNECTED BUT THOSE PIECES WILL LOOK LIKE (*). CALL THESE PIECES $i$-STRINGS OR ROOT STRINGS. THE REFLECTION $\sigma_i$ ON $\Lambda$ HAS THE EFFECT

$$\mu \to \mu - \langle wt(\mu), \alpha_i \rangle \alpha_i$$

WE CAN TRANSLATE THIS INTO AN OPERATION ON THE "ROOT STRING (*)"

$$\begin{array}{ccccccccccc}
\sigma_i(f) & & \sigma_i(e) & & & & & & & & \sigma_i(a) \\
\shortparallel & & \shortparallel & & & & & & & & \shortparallel \\
a & & b & & c & & d & & e & & f \\
\bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet
\end{array}$$

FLIP END TO END.

THIS HAS THE FOLLOWING EFFECT.

REMEMBER

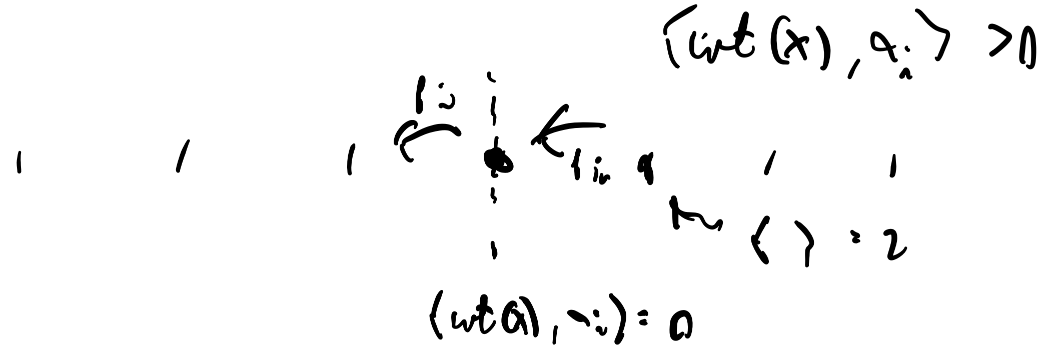
$$wt(e_i(x)) = wt(x) + \alpha_i$$

$$wt(f_i(x)) = wt(x) - \alpha_i$$

$$\boxed{\langle wt(x), \alpha_i \rangle = \varphi_i(x) - \varepsilon_i(x)}$$

DEFINE

$$\Delta_i(x) = \begin{cases} f_i^{\langle wt(x), \alpha_i \rangle}(x) & \text{IF } wt(x) \geq 0 \\ e_i^{-\langle wt(x), \alpha_i \rangle}(x) & \end{cases}$$



$\Delta_i \Delta_j = \Delta_j \Delta_i$ IF $|i-j| > 1$

$\Delta_i^2 = \mathrm{ID}$.

$\Delta_i \Delta_j \Delta_i = \Delta_j \Delta_i \Delta_j$ IF $j = i \pm 1$.

CORRESPONDING ACTIONS ON TABLEAUX WERE DEFINED LONG AGO

ACTION OF $S_n$ ON TABLEAUX.

EITHER BENDER AND KNUTH OR LASCOUX AND SCHÜTZENBERGER.

$$\mathrm{wt}(\Delta_i(X)) = \Delta_i(\mathrm{wt}(X)).$$

EXAMPLE: SOME GL(2) CRYSTALS.

ONLY ONE ROOT $i = 1$ $\quad \alpha = (1,-1)$

THE IRREDUCIBLE CRYSTALS ARE

$$\det^{k} \otimes \mathrm{Sym}^{r}(\mathbb{C}^2) = \det(k) \otimes V^{r}(\mathbb{C}^2)$$

OMIT THE DETERMINANT FACTOR.

$V^3(\mathbb{C})$

$$\boxed{2\,|\,2\,|\,2} \xleftarrow{1} \boxed{1\,|\,2\,|\,2} \xleftarrow{1} \boxed{1\,|\,1\,|\,2} \xleftarrow{1} \boxed{1\,|\,1\,|\,1}$$

$(0,3) \qquad (1,2) \qquad (2,1) \qquad (3,0)$

WEIGHTS

$-3 \qquad -1 \qquad 1 \qquad 3 \qquad \langle \alpha \;\; wt(x) \rangle$

$$s_i(x) = f_i^k(x) \quad \text{IF} \quad \langle \alpha_i, wt(x) \rangle = k \geq 0$$

$$e_i^l(x) \quad \langle \quad \rangle = -l \leq 0$$

$$s_1 \boxed{1\ 1\ 1} = f_1^3\left(\boxed{1\ 1\ 1}\right) = \boxed{2\ 2\ 2}$$

$$\langle wt(x), \alpha_i \rangle = \varphi_i(x) - \varepsilon_i(x)$$

IMPLIES IF $\{x\}$ IS A SINGLETON ROOT STRING $\varphi_i(x) = \varepsilon_i(x) = 0$ THEN

$$\langle wt(x), \alpha_i \rangle = 0$$

CRYSTAL OF $\Pi_{(2,1,0)}$ (ADJOINT SQUARE)

ELEMENTS ARE SSYT OF SHAPE $(2,1)$

$$
\begin{tikzcd}
 & \boxed{\begin{smallmatrix}1&1\\2&\end{smallmatrix}} \arrow[dl, "1"'] \arrow[dr, "2"] & & \\
\begin{smallmatrix}1&2\\2&\end{smallmatrix} \arrow[d, "2"] & & \boxed{\begin{smallmatrix}1&1\\3&\end{smallmatrix}} \arrow[d, "1"] & wt = (2,0,1) \\
\begin{smallmatrix}1&3\\2&\end{smallmatrix} \arrow[d, "2"] & & \begin{smallmatrix}1&2\\3&\end{smallmatrix} \arrow[d, "1"] & \\
\begin{smallmatrix}1&3\\3&\end{smallmatrix} \arrow[dr, "1"'] & & \begin{smallmatrix}2&2\\3&\end{smallmatrix} \arrow[dl, "2"] & \\
 & \begin{smallmatrix}2&3\\3&\end{smallmatrix} & &
\end{tikzcd}
$$

$wt(T) = (\mu_1, \mu_2, \cdots, \mu_n)$

$\mu_i$ = # OF $i$'S IN THE TABLEAU T.

$f_i(T)$ TURNS SOME $i$ ENTRY TO $i+1$

$e_i(T)$ $i+1$ TO $i$.

KASHIWARA PUT A CRYSTAL STRUCTURE ON THE SET $\mathcal{B}_\lambda$ OF SSYT OF SHAPE $\lambda$ FOR $\lambda$ A PARTITION.

MANY TABLEAUX IDEAS WE KNOWN BEFORE BY YOUNG ROBINSON, D.E. LITTLEWOOD, SCHENSTED, KNUTH, BENDER, LASCOUX, SCHÜTZENBERGER.